CHARACTERISTICS OF TURBULENT-EXCHANGE COEFFICIENTS IN A VISCOUS UNDERLAYER

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The transient equations for longitudinal (along the flow) velocity pulsations and pulsations of concentration in the region of a viscous underlayer are considered. Estimates based on experimental data enable the contributions arising from certain terms of these equations to be neglected in turbulent transfer. Subject to this approximation, expressions are obtained for the turbulent viscosity and diffusion coefficients. These coefficients behave differently in the case of large Schmidt (Prandtl) numbers. The behavior of the turbulent Schmidt number in the region of the viscous underlayer is analyzed.

1. It follows from earlier experiments [1, 2] that the longitudinal component of pulsational velocity u in a viscous underlayer varies linearly with distance from the wall ($u \sim y$). This relationship is also valid for the tangential velocity pulsations ($w \sim y$). For the component perpendicular to the wall we obtain the following expression from the continuity equation:

$$v = -\int\limits_0^y \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) dy \sim y^2 \tag{1.1}$$

for the turbulent Reynolds stresses we have $\langle uv \rangle \sim y^3$ and for the turbulent viscosity

$$\nu_T = -\langle uv \rangle (dU/dy)^{-1} \sim y^3 \tag{1.2}$$

where U (y) is the mean flow velocity. Here and subsequently the angular brackets signify time-averaging (or averaging over a statistical assembly).

From qualitative considerations, Landau and Levich proposed the following equation for turbulent viscosity in the region of a viscous underlayer

$$\nu_T = \langle v^2 \rangle T \sim y^4 \tag{1.3}$$

where T is the average period of the velocity pulsations, a quantity which is independent of y by virtue of the linearity of the equations of motion in the viscous underlayer [3]. Sometimes it has been pointed out that Eq. (1.2) does not allow for the possibility of a change in the correlation coefficient with distance from the wall, i.e., it is assumed that

$$K_{uv} = \langle uv \rangle / \sqrt{\langle u^2 \rangle \langle v^2 \rangle} \sim y$$

We note that, for the case of two-dimensional turbulence, depending solely on x and y, if we assume that the system is homogeneous with respect to x there is an exact proof of the fact that $K_{uv} = 0$ for $y = 0$.

From the continuity equations (1.1), confining attention to the first nonvanishing terms, we have

$$v = -\frac{\partial u'}{\partial x}\frac{y^2}{2}, \quad u' = \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\langle uv \rangle = -\left\langle u' \frac{\partial u'}{\partial x} \right\rangle \frac{y^3}{2} = -\frac{\partial}{\partial x}\langle u'^2 \rangle \frac{y^3}{4}$$

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In view of the statistical homogeneity with respect to x, the quantity $\langle u'^2 \rangle$ is independent of x, so that the coefficient of the third power in the expansion of the Reynolds stresses in powers of y is equal to zero, and the expansion starts with the term in y^4.

Experiments [4] show, however, that turbulence in the viscous underlayer had a considerable three-dimensional component, so that the proof based on the two-dimensional assumption fails, and the question as to the validity of Eqs. (1.2) and (1.3) remains open. We tried to establish the turbulent-exchange attenuation power index by making use of existing data relating to mass transfer at high Schmidt numbers ($S = \nu/D \gg 1$ where ν is the viscosity and D the diffusion coefficient), for which the diffusion layer is completely "submerged" in the viscous underlayer and turbulent pulsations play a decisive role in mass transfer. However, the accuracy of these experiments is insufficient, since they admit interpretations in favor of both third [5-7] and fourth [8-10] power laws for the attentuation.

It was assumed in the papers cited [3, 5-10] that the behavior of the turbulent viscosity and diffusion coefficients was identical. This is equivalent to the assertion that the turbulent Schmidt number is constant over the thickness of the viscous underlayer ($S_T = \nu_T/D_T = \mathrm{const}$). However, for $S \gg 1$ this assumption is not satisfied. This was pointed out in [11], and a scheme was proposed according to which $\nu_T \sim y^3$, $D_T \sim y^4$ in the viscous underlayer, and hence the turbulent Schmidt number

$$S_T \sim y^{-1} \tag{1.4}$$

In this paper the equation in question is obtained for the limiting case as $S \to \infty$.

2. Let us consider the turbulent flow of an incompressible liquid flowing in the direction of the x axis over a plane surface (y=0). U (y) is the average flow velocity, u, v, w are the pulsational velocity components in the directions x, y, z, and p are the pressure pulsations. All the variables (dependent and independent) are made dimensionless by means of the viscosity ν and the dynamic velocity $v_* = \sqrt{\tau_W/\rho}$, where τ_W is the friction at the wall, ρ is the density of the liquid. The turbulence is assumed homogeneous in the x and z directions and steady in time.

In dimensionless form the equation for u appears thus

$$\frac{\partial u}{\partial t} - \Delta u = -v\frac{dU}{dy} - Q$$
$$Q = U\frac{\partial u}{\partial x} + \frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv - \langle uv \rangle) + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} \tag{2.1}$$

where Δ is the Laplace operator.

Considering the right-hand side in Eq. (2.1) as an inhomogeneity, we may rewrite it in integral form, allowing for the "adhesion" boundary condition $(u)_{y=0} = 0$

$$u(\mathbf{x}, t) = -\iiint\limits_{y' \geqslant 0} d\mathbf{x}' \int\limits_{-\infty}^{t} dt' \left[v(\mathbf{x}', t')\frac{dU(y')}{dy'} + Q(\mathbf{x}', t')\right] G(\mathbf{x}, \mathbf{x}'; t - t') \tag{2.2}$$

where G (**x**, **x'**; t−t') is the Green's function of the heat-conduction equation

$$G(\mathbf{x}, \mathbf{x}'; t - t') = \exp\left(-\frac{(x - x')^2 + (z - z')^2}{4(t - t')}\right)[4\pi(t - t')]^{-3/2}\left[\exp\left(-\frac{(y - y')^2}{4(t - t')}\right) - \exp\left(-\frac{(y + y')^2}{4(t - t')}\right)\right] \tag{2.3}$$

In Eq. (2.2) we replace the integration variable $t' = t - \tau$ after which we multiply (2.2) by v (**x**, t) and average. We obtain the Reynolds stresses

$$\langle uv \rangle = -\iiint\limits_{y' \geqslant 0} d\mathbf{x}' \int\limits_{0}^{\infty} d\tau \left(\langle vv' \rangle \frac{dU'}{dy'} + \langle vQ' \rangle\right) G(\mathbf{x}, \mathbf{x}'; \tau) \tag{2.4}$$

where the prime on the functions in the brackets means that these depend on the integration variables x' and τ.

We may neglect the term $\langle vQ' \rangle$ in the integrand by comparison with the first term in round brackets. The following grounds exist for this:

1) The nonlinear terms in Q are small in the region of the viscous underlayer, and in the Reynolds stresses (2.4) they give triple correlation functions of the velocity pulsations, which are much smaller than the second correlation $\langle vv' \rangle$.

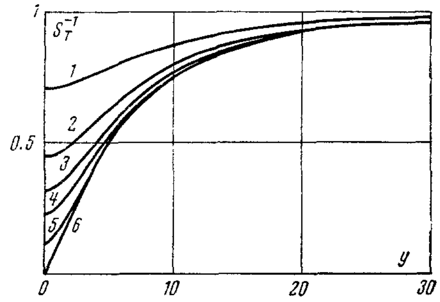

Fig. 1

2) Experiments [4] aimed at visualizing the flow in the viscous underlayer showed that the wall eddies were greatly extended along the flow; this may be explained by the action of the mean velocity gradient dU/dy. The mean extent of the eddies in the z direction is $\Delta z \sim 10^2$, the extent along the x axis is, on average, an order of magnitude greater ($\Delta x \sim 10^3$) [12, 13, 4], so that the three terms in Q containing derivatives with respect to x may be neglected, since the amplitudes of the velocity pulsations are of the order of unity over the whole turbulent region, and in the region of the underlayer diminish as the corresponding powers of y.

3) As $y \to 0$ all the terms on the right-hand side of Eq. (2.1) tend to zero except $\partial p/\partial x$; the amplitude of the pressure pulsations is small ($\sqrt{\langle p^2\rangle} \sim 1$) [14] while the scale of the eddies along x is great ($\Delta x \sim 10^3$). Hence the layer in which $|\partial p/\partial x| > |v| dU/dy$ is thin and makes a negligibly small contribution to the Reynolds stresses.

Thus the term proportional to the mean velocity gradient makes the chief contribution to the Reynolds stresses.

By virtue of its homogeneity and steady-state nature, the correlation $\langle vv'\rangle$ depends solely on y, y', x−x', z−z', and $\tau = t-t'$. The mean period of the pulsations of the viscous underlayer was determined experimentally in [13] ($T \sim 20$). If the function $\langle vv'\rangle$ falls rapidly for $\tau = t-T' > 20$, then analysis of Eqs. (2.3) and (2.4) shows that the dependence on the slowly-changing arguments x − x' and z − z' in the function may be neglected (remembering that $\Delta x \sim 10^3$, $\Delta z \sim 10^2$). This corresponds to neglecting the second derivatives with respect to x and z in the Laplacian of Eq. (2.1) (boundary-layer approximation). After integrating with respect to x' and z' in (2.4) we obtain

$$\langle uv\rangle = -\int_0^\infty dy' \int_0^\infty d\tau \langle v(y,t)\, v(y', t-\tau)\rangle G(y, y'; \tau) \frac{dU(y')}{dy'} = -\nu_T \frac{dU}{dy} \tag{2.5}$$

where the mean velocity gradient is taken outside the integration sign, being a function which varies only slowly in the viscous underlayer. For the turbulent viscosity we obtain from (2.5)

$$\nu_T(y) = \int^\infty dy' \int_0^\infty d\tau R(y, y'; \tau) G(y, y'; \tau), \quad R(y, y'; \tau) = \langle v(y,t)\, v(y', t-\tau)\rangle \tag{2.6}$$

where $G(y, y'; \tau)$ is the Green's function of the heat-conduction equation on a semiinfinite straight line

$$G(y, y'; \tau) = (4\pi\tau)^{-1/2}\left[\exp\left(-\frac{(y-y')^2}{4\tau}\right) - \exp\left(-\frac{(y+y')^2}{4\tau}\right)\right] \tag{2.7}$$

Since $G \sim y$ and $R \sim y^2$ as $y \to 0$, Eq. (2.6) leads to a third-power attenuation law: $\nu_T \sim y^3$.

3. In dimensionless form the equation for the pulsations of concentration c appears thus

$$\frac{\partial c}{\partial t} - \frac{1}{S}\Delta c = -v\frac{dC}{dy} - U\frac{\partial c}{\partial x} - \frac{\partial uc}{\partial x} - \frac{\partial}{\partial y}(vc - \langle vc\rangle) - \frac{\partial wc}{\partial z} \tag{3.1}$$

where C (y) is the mean concentration.

The boundary condition for Eq. (3.1) is the same as in the previous case $[(c)_{y=0} = 0]$. The problem is entirely analogous to the case set out in section 2. The approximation used there is valid here also; in fact in the present case it has better grounds since no pressure gradient appears in (3.1). When the Schmidt number $S \gg 1$ it may be shown (using perturbation theory) that the rejected last four terms in (3.1) make a contribution to the turbulent mass flow $\langle vc\rangle$ proportional to higher powers of y than the retained term vdC/dy.

Omitting the calculations, let us simply give the final expression for the turbulent diffusion coefficient

$$D_T(y) = \int_0^\infty dy' \int_0^\infty d\tau R(y, y'; \tau) G(y, y'; \tau/S) \tag{3.2}$$

which conicides with Eq. (2.6) except for the time argument in the Green's function (2.7), which in the pre-present case is divided by S.

Analysis of Eq. (3.2) shows that for large but finite S there is a narrow region $y < S^{-1/2}$ in which D $D_T \sim y^3$. If we let S tend to infinity and use the Green's function property

$$\lim_{\alpha \to 0} G(y, y'; \alpha) = \delta(y - y')$$

where $\delta(y-y')$ is the Dirac delta function, we obtain the Landau–Levich equation (1.3), which is only valid for turbulent diffusion when $S \gg 1$, but not for turbulent viscosity

$$D_T(y) = \int_0^\infty \langle v(y, t) v(y, t-\tau) \rangle d\tau = \langle v^2 \rangle T \sim y^4$$

where the mean period of the pulsations T is the Lagrange time scale

$$T = \int_0^\infty R(y, y; \tau) d\tau / R(y, y; 0)$$

4. For the correlation function R in the viscous underlayer we may propose the following approximation:

$$R(y, y'; \tau) = A y^2 y'^2 e^{-\tau/T}$$

where A and T are constants. Then the integrals in Eqs. (2.6) and (3.2) may be expressed in elementary functions; after dividing (2.6) by (3.2) we obtain the turbulent Schmidt number

$$S_T = S \frac{1 - \exp(-y/\sqrt{T}) + y^2/2T}{1 - \exp(-y\sqrt{S/T}) + y^2 S/2T} \tag{4.1}$$

We see from (4.1) that $S_T \equiv 1$ for $S=1$. This may also be concluded from Eqs. (2.6) and (3.2). For $S \gg 1$ there are three regions of differing S_T behavior: $y < \sqrt{T/S}$ $S_T \sim \sqrt{S}$ in the region $\sqrt{T} > y > \sqrt{T/S}$ $S_T \sim y^{-1}$; for $y > \sqrt{T}$ the turbulent Schmidt number is close to unity, and in the limit $(y \to \infty)$ we have $S_T = 1$. Figure 1 shows the behavior of the turbulent Schmidt number (S_T^{-1}) for $T=20$ and $S=2, 5, 10, 20, 80, \infty$ (curves 1, 2, 3, 4, 5, 6 respectively).

Data regarding the turbulent Schmidt number are extremely contradictory, but there is not doubt as to the fact that on moving away from the wall it ceases to depend on molecular effects (on S) and becomes constant. In this case $S_T \to 1$ as $y \to \infty$. This is a consequence of the approximation taken (the convective terms and $\partial p/\partial x$ are rejected), which is invalid outside the viscous underlayer.

5. If motion of the liquid takes place close to the wall with a characteristic time T, the dimensions of the regions in which the influence of viscosity and diffusion is felt (from dimensional considerations) amount to

$$y_1 \sim \sqrt{\nu T}, \quad y_2 \sim \sqrt{DT} \tag{5.1}$$

where all the quantities are dimensional.

For $\nu \gg D$ there is a region $y_1 \gg y \gg y_2$ in which we may neglect the action of molecular diffusion on turbulent (molar) mass transfer, but we must still not neglect the action of viscosity on the velocity pulsations, and hence on the turbulent transfer of momentum. Neglect of molecular effects leads to Eq. (1.3). Equation (5.1) in dimensionless form gives those values of the coordinate y for which the behavior of S_T changes in Eq. (4.1).

If heat is regarded as a passive impurity, all the results are also valid for heat-transfer processes; it is only necessary to replace the concentration by temperature and the Schmidt number by the Prandtl number Pr.

We may therefore reasonably assume that the law of turbulent friction (1.2) holds in the viscous underlayer, and the limiting value of the turbulent Schmidt (Prandtl) number is given by the equation

$$\lim_{S,\, Pr \to \infty} (S_T,\, Pr_T) \sim y^{-1} \quad (1.4)^*$$

Thus in experiments on mass transfer with large Schmidt numbers only the behavior of D_T (y) may be determined; this relationship should approach the fourth-power law of attenuation move closely, the greater the value of S. The behavior of ν_T (y) cannot be determined from these experiments.

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* Equation number appears as in Russian original – Publisher.